Pergamon

Int. J. Solids Structures Vol. 33, No. 7, pp. 977–990, 1996

0020–7683/96 \$15.00 + .00

0020–7683(95)00083–6

EXACT ELECTROELASTIC ANALYSIS OF PIEZOELECTRIC LAMINAE VIA STATE SPACE APPROACH

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(*Received 21 October 1994; in revised form 24 March 1995*)

Abstract—In an attempt to develop an efficient analytical approach for the electromechanical analysis of laminated piezoelectric structures, an exact transfer-matrix-based methodology is presented. The state space equations for a three-dimensional piezoelectric lamina are first derived by eliminating the in-plane stresses and electric displacements from the governing equations. The transfer matrix is rendered in block form by judiciously arranging the state variables and its property is then exploited to minimize the computational effort. Using the approach, an exact analysis of coupled electroelastic behavior of a rectangular piezoelectric plate of 6mm crystal symmetry subjected to a mechanical or electrical load is presented. A simply supported square plate made of barium titanate is analyzed as a numerical example and results are compared to those from the corresponding uncoupled analysis.

1. INTRODUCTION

With the recent development of piezoelectric ceramics and polymers and associated technological applications, mechanical behavior of piezoelectric structural systems has been the topic of numerous investigations [see Tzou and Anderson (1992); Lee *et al.* (1993); and Rao and Sunar (1994) for extensive references]. Recent studies on piezoelectric materials have resulted in qualitative as well as quantitative understanding of interactions between the mechanical and electric fields in infinite and semi-infinite piezoelectric media with or without crack-like defects or inclusions (Pak, 1990; Wang, 1992; Chen, 1993; Dunn, 1994; Lee and Jiang, 1994a; Sosa and Castro, 1994). However, electroelastic behavior of laminated piezoelectric composite structures has received very little attention despite the fact that many piezoelectric devices are constructed in laminated form as in the case of piezoelectric resonators, multilayer capacitors (MLC), or multilayer actuators (MLA). In an MLA, for example, which consists of layered piezoelectric ceramics with interconnected electrodes, electrode migration and internal stresses are of great concern as a result of high operating electric fields and strains associated with actuation. To date, however, models of strain actuator and substrate systems are very limited because most of the studies have been focused on the implementation of control algorithms. Most of the models in the literature are based on either the Kirchhoff hypothesis or shear-deformable plate theories which do not account for the continuity of interlaminar shear stresses. The lack of more exact models or solution methods for laminated piezoelectric structures is the direct motivation for the study presented herein.

Recently, it was suggested that the state space approach be used for the analysis of laminated piezoelectric structures (Sosa, 1992; Jiang and Lee, 1993). Based on the mixed formulation of elasticity, the state space approach converts a boundary value problem to an equivalent initial value problem in terms of state variables (Bahar, 1977). Once the transfer matrix for each layer is found, a global matrix can be assembled by introducing interlayer contact and boundary conditions. The order of the global matrix does not depend on the number of layers since the matrix is multiplicative in nature for certain interlayer contact conditions (Lee and Jiang 1994b). Therefore, once the transfer matrix of a single layer or lamina is obtained, any arbitrary lamination of layers can be accommodated by taking advantage of the multiplicative nature of the transfer matrix.

Based on the linear theory of piezoelasticity, a state space approach for exact analysis of three-dimensional piezoelectric lamina is presented in this paper with the aim at developing an efficient analytical methodology for laminated piezoelectric structures. Since the eight state variables appearing in the state equations directly represent the boundary conditions on the upper and lower surfaces, any arbitrary loading conditions can be taken into account. All physical quantities associated with the problem (i.e. stresses and electric displacements as well as mechanical displacements and electrical potential) can be solved directly from the state equations. Using the state space approach developed in the study, an exact solution is obtained for a simply supported rectangular thick lamina in the form of infinite series. This solution can be evaluated to any desired level of accuracy by retaining an appropriate number of series terms. A notable feature of the solution approach presented herein is that the computational task is significantly reduced by rendering the transfer matrix in block form and exploiting the symmetry properties of the block matrices.

In contrast to some existing formulations on electroelasticity which collapse in the case of pure elasticity, the present formulation can recover as a special case pure elasticity so that pure elasticity and piezoelectricity can be studied on equal footing. This makes it possible to do a direct comparison between the two theories in detail, thereby revealing intricate mechanoelectric coupling effects. Moreover, the same approach can be used for non-piezoelectric layers (e.g. structural layers and electrodes) especially when modeling multiply-laminated piezoelectric structures such as MLAs. A square barium titanate plate with simple supports is analyzed as a numerical example and results are compared to those from the corresponding uncoupled analysis. It should be noted that the solution presented in this study is a full three-dimensional solution and preserves the elastic and electric anisotropy as well as the full electromechanical coupling.

2. EQUATIONS OF PIEZOELECTRICITY

Here, we present a brief summary of linear piezoelasticity. More general exposition of nonlinear piezoelasticity can be found elsewhere (e.g. Eringen and Maugin, 1991). The description of piezoelectricity is based on the combination of elements of elasticity and electrodynamics, where the basic variables are the stress $\boldsymbol{\sigma}$, and strain $\mathbf{s}$, the electric displacement $\mathbf{D}$, and the electric field $\mathbf{E}$. If body forces and electric charge density are ignored, the electroelastic field is governed by

$$\nabla \cdot \boldsymbol{\sigma} = 0, \quad \nabla \cdot \mathbf{D} = 0, \tag{1}$$

where the first equation is the equilibrium equation of elasticity and the second is Gauss's law of electrostatics. The elastic displacement $\mathbf{u}$ and the electric potential ϕ are introduced as follows:

$$\mathbf{s} = \tfrac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla), \quad \mathbf{E} = -\nabla\phi, \tag{2}$$

where the second expression implies the quasistatic approximation, viz. $\nabla \times \mathbf{E} = 0$. In the above equations, only 13 relations exist for 22 unknowns. The additional nine equations are provided by the constitutive relations reflecting electro-elastic coupling effects. These relations can be derived from thermodynamics potentials (Tiersten, 1969) and can be cast into four different forms depending on the choice of independent variables. If the strains and electric field are chosen as the independent variables, the constitutive equations take the following form:

$$\boldsymbol{\sigma} = \mathbf{C}\mathbf{s} - \mathbf{e}^{T}\mathbf{E} \tag{3a}$$

$$\mathbf{D} = \mathbf{e}\mathbf{s} + \boldsymbol{\varepsilon}\mathbf{E}, \tag{3b}$$

where $\mathbf{C}$ is the fourth order tensor of elastic moduli measured at constant or zero electric field, $\mathbf{e}$ the third-order piezoelectric tensor, $\boldsymbol{\varepsilon}$ the second-order dielectric tensor measured at constant or zero strain. The generalized Hooke's law for pure elasticity and the constitutive description of rigid dielectrics can be recovered by setting the piezoelectric constants **e** equal to zero.

For common piezoelectric materials of 6mm crystal symmetry, the constitutive relations (3) can be written as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} s_{11} \\ s_{22} \\ s_{33} \\ s_{23} \\ s_{31} \\ s_{12} \end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix} \tag{4a}$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} s_{11} \\ s_{22} \\ s_{33} \\ s_{23} \\ s_{31} \\ s_{12} \end{Bmatrix} + \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}, \tag{4b}$$

where $C_{66} = C_{11} - C_{12}$. Constitutive relations of other classes of crystal symmetry can be found elsewhere (Nye, 1976). It is noted that most man-made piezoelectric materials commonly used for fabrication of multi-layered electromechanical devices can be represented by the 6mm symmetry class.

3. STATE SPACE FORMULATION

The state space approach which has its roots in classical dynamics has been widely used in modern control theory. Based on the mixed formulation of elasticity proposed by Vlasov and Leontev (1966), this approach has also been utilized by several authors to treat problems in elasticity (Bahar, 1977; Steele and Kim, 1990; Lee and Jiang, 1994b). Recently, Sosa and Castro (1994) applied the state space methodology to obtain a point-force solution for a piezoelectric half-space. Unlike in control theory, however, derivation of the state space equation from the field equations is not always straightforward in the case of elasticity or piezoelectricity. Since the field equations of piezoelectricity involve variables that cannot be prescribed on the boundary, elimination of these variables is requisite to obtain the state space equation. Following the process of state space approach in elasticity, the field equation can be recast in the following matrix form by eliminating the in-plane stresses σ_{11}, σ_{22} and σ_{12} as well as the in-plane electric displacements D_1 and D_2 from the governing equations (1)–(3):

$$\frac{\partial}{\partial x_3}\mathbf{X}(x_1, x_2, x_3) = \begin{bmatrix} 0 & \mathbf{T}_1 \\ \mathbf{T}_2 & 0 \end{bmatrix} \mathbf{X}(x_1, x_2, x_3) \tag{5}$$

and

$$\mathbf{Y}(x_1,x_2,x_3)=\begin{bmatrix}\mathbf{Q}_1 & 0\\ 0 & \mathbf{Q}_2\end{bmatrix}\mathbf{X}(x_1,x_2,x_3), \tag{6}$$

where

$$\mathbf{X}(x_1,x_2,x_3)=\{U\ \ V\ \ \sigma_{33}\ \ D_3\ \ \sigma_{13}\ \ \sigma_{23}\ \ W\ \ \phi\}^{\mathrm{T}} \tag{7}$$

$$\mathbf{Y}(x_1,x_2,x_3)=\{\sigma_{11}\ \ \sigma_{12}\ \ \sigma_{22}\ \ D_1\ \ D_2\}^{\mathrm{T}} \tag{8}$$

$$\mathbf{T}_1=\begin{bmatrix}\dfrac{1}{C_{44}} & 0 & -\partial_1 & -\dfrac{e_{15}}{C_{44}}\partial_1\\ 0 & \dfrac{1}{C_{44}} & -\partial_2 & -\dfrac{e_{15}}{C_{44}}\partial_2\\ -\partial_1 & -\partial_2 & 0 & 0\\ -\dfrac{e_{15}}{C_{44}}\partial_1 & -\dfrac{e_{15}}{C_{44}}\partial_2 & 0 & \left(\varepsilon_{11}+\dfrac{e_{15}^2}{C_{44}}\right)(\partial_1^2+\partial_2^2)\end{bmatrix} \tag{9}$$

$$\mathbf{T}_2=\begin{bmatrix}-A_1\partial_1^2-C_{66}\partial_2^2 & -(A_2+C_{66})\partial_{12}^2 & -A_4\partial_1 & -A_3\partial_1\\ -(A_2+C_{66})\partial_{12}^2 & -C_{66}\partial_1^2-A_1\partial_2^2 & -A_4\partial_2 & -A_3\partial_2\\ -A_4\partial_1 & -A_4\partial_2 & A\varepsilon_{33} & Ae_{33}\\ -A_3\partial_1 & -A_3\partial_2 & Ae_{33} & -AC_{33}\end{bmatrix} \tag{10}$$

$$\mathbf{Q}_1=\begin{bmatrix}A_1\partial_1 & A_2\partial_2 & A_4 & A_3\\ C_{66}\partial_2 & C_{66}\partial_1 & 0 & 0\\ A_2\partial_1 & A_1\partial_2 & A_4 & A_3\end{bmatrix} \tag{11}$$

$$\mathbf{Q}_2=\begin{bmatrix}\dfrac{e_{15}}{C_{44}} & 0 & 0 & -\left(\varepsilon_{11}+\dfrac{e_{15}^2}{C_{44}}\right)\partial_1\\ 0 & \dfrac{e_{15}}{C_{44}} & 0 & -\left(\varepsilon_{11}+\dfrac{e_{15}^2}{C_{44}}\right)\partial_2\end{bmatrix}, \tag{12}$$

with

$$\begin{aligned}A_1&=(C_{11}e_{33}^2+C_{11}C_{33}\varepsilon_{33}-C_{13}^2\varepsilon_{33}-2C_{13}e_{31}e_{33}+C_{33}e_{31}^2)A\\ A_2&=(C_{12}e_{33}^2+C_{12}C_{33}\varepsilon_{33}-C_{13}^2\varepsilon_{33}-2C_{13}e_{31}e_{33}+C_{33}e_{31}^2)A\\ A_3&=(C_{13}e_{33}-e_{31}C_{33})A\\ A_4&=(C_{13}\varepsilon_{33}+e_{31}e_{33})A\\ A&=(e_{33}^2+C_{33}\varepsilon_{33})^{-1},\end{aligned} \tag{13}$$

where $\partial_i=\partial/\partial x_i$.

The above eqn (5) is the state space equation for piezoelasticity where eight unknowns σ_{13}, σ_{23}, σ_{33}, D_3, U, V, W and ϕ are chosen as the state variables, where $\mathbf{u}=\{U,V,W\}$. It should be noted that the block structure of the coefficient matrix in (5) is not a coincidence but, in fact, a product of judiciously ordering the state variables in order to minimize the computational effort as will be shown below. It should also be noted that the state space equation for piezoelasticity is structurally the same as the state equation for pure elasticity (Iyengar and Pandya 1983) except for the two additional quantities due to the electric field contribution. The state space equation for pure elasticity can be recovered by setting the piezoelectric constant, e_{ij}, to zero. In such a case, the mechanical quantities are independent of the electrical quantities since the coupling coefficients in the transfer matrix vanish.

4. EXACT SOLUTION FOR RECTANGULAR PIEZOELECTRIC PLATE

Here we employ the state space equation to find an exact solution for a rectangular plate which is simply supported on all four edges. The edge conditions are given by:

$$
\begin{aligned}
\sigma_{11} &= V = W = \phi = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad a \\
\sigma_{22} &= U = W = \phi = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad b.
\end{aligned} \tag{14}
$$

The state variables which exactly satisfy the boundary conditions can be written by

$$
\begin{aligned}
U(x,y,z) &= \sum_m \sum_n \bar{U}_{mn}(z) \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\
V(x,y,z) &= \sum_m \sum_n \bar{V}_{mn}(z) \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \\
W(x,y,z) &= \sum_m \sum_n \bar{W}_{mn}(z) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\
\phi(x,y,z) &= \sum_m \sum_n \bar{\phi}_{mn}(z) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\
\sigma_{33}(x,y,z) &= \sum_m \sum_n \bar{\sigma}_{33mn}(z) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\
\sigma_{13}(x,y,z) &= \sum_m \sum_n \bar{\sigma}_{13mn}(z) \cos\frac{m\pi x}{a} \sin\frac{n\pi x}{b} \\
\sigma_{23}(x,y,z) &= \sum_m \sum_n \bar{\sigma}_{23mn}(z) \sin\frac{m\pi x}{a} \cos\frac{n\pi x}{b} \\
D_3(x,y,z) &= \sum_m \sum_n \bar{D}_{3mn}(z) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}.
\end{aligned} \tag{15}
$$

Substituting the above expressions into (5) leads to the following matrix equation:

$$
\frac{\mathrm{d}}{\mathrm{d}z}\bar{\mathbf{S}}(z) = \begin{bmatrix} 0 & \mathbf{P} \\ \mathbf{Q} & 0 \end{bmatrix} \bar{\mathbf{S}}(z), \tag{16}
$$

where

$$
\bar{\mathbf{S}} = \{\bar{U} \quad \bar{V} \quad \bar{\sigma}_{33} \quad \bar{D}_3 \quad \bar{\sigma}_{13} \quad \bar{\sigma}_{23} \quad \bar{W} \quad \bar{\phi}\}^{\mathrm{T}} \tag{17}
$$

$$
\mathbf{P} = \begin{bmatrix}
\dfrac{1}{C_{44}} & 0 & -\xi & -\dfrac{e_{15}}{C_{44}}\xi \\
0 & \dfrac{1}{C_{44}} & -\eta & -\dfrac{e_{15}}{C_{44}}\eta \\
\xi & \eta & 0 & 0 \\
\dfrac{e_{15}}{C_{44}}\xi & \dfrac{e_{15}}{C_{44}}\eta & 0 & -\left(\varepsilon_{11} + \dfrac{e_{15}^2}{C_{44}}\right)(\xi^2+\eta^2)
\end{bmatrix} \tag{18}
$$

$$\mathbf{Q} = \begin{bmatrix} A_1\xi^2 + C_{66}\eta^2 & (A_2 + C_{66})\xi\eta & -A_4\xi & -A_3\xi \\ (A_2 + C_{66})\xi\eta & C_{66}\xi^2 + A_1\eta^2 & -A_4\eta & -A_3\eta \\ A_4\xi & A_4\eta & A\varepsilon_{33} & Ae_{33} \\ A_3\xi & A_3\eta & Ae_{33} & -AC_{33} \end{bmatrix}, \tag{19}$$

where $\xi = m\pi/a$ and $\eta = n\pi/b$. Note that in the above equations the subscripts m and n are dropped for brevity.

The solution to (16) can be written as (Gantmacher, 1960):

$$\bar{\mathbf{S}}(z) = \exp[z\mathbf{K}]\bar{\mathbf{S}}(0) = \mathbf{T}\bar{\mathbf{S}}(0), \tag{20}$$

where the exponential matrix $\mathbf{T} = \exp[z\mathbf{K}]$ is the transfer matrix that propagates the initial state vector on the lower surface into the field point at coordinate z. At this point, it remains to evaluate the transfer matrix explicitly. The evaluation can be done by following the approach described in Jiang and Lee (1993). In view of the considerable computational task involved due to the size of the transfer matrix (8×8), however, a more efficient alternative procedure shall be sought to reduce the computational labor in this study. We shall show in the following that only a 4×4 matrix needs to be evaluated for eigenvalues instead of the original 8×8 matrix. Noting the block structure of the transfer matrix, we rewrite (16) as

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{Bmatrix} \bar{\mathbf{R}}_1(z) \\ \bar{\mathbf{R}}_2(z) \end{Bmatrix} = \begin{bmatrix} 0 & \mathbf{A} \\ \mathbf{B} & 0 \end{bmatrix}\begin{Bmatrix} \bar{\mathbf{R}}_1(z) \\ \bar{\mathbf{R}}_2(z) \end{Bmatrix}, \tag{21}$$

where

$$\begin{aligned} \bar{\mathbf{R}}_1 &= \{\bar{U} \quad \bar{V} \quad \bar{\sigma}_{33} \quad \bar{D}_3\}^{\mathrm{T}} \\ \bar{\mathbf{R}}_2 &= \{\bar{\sigma}_{13} \quad \bar{\sigma}_{23} \quad -\bar{W} \quad -\bar{\phi}\}^{\mathrm{T}} \end{aligned} \tag{22}$$

$$\mathbf{A} = \begin{bmatrix} \dfrac{1}{C_{44}} & 0 & -\xi & -\dfrac{e_{15}}{C_{44}}\xi \\ 0 & \dfrac{1}{C_{44}} & -\eta & -\dfrac{e_{15}}{C_{44}}\eta \\ -\xi & -\eta & 0 & 0 \\ -\dfrac{e_{15}}{C_{44}}\xi & -\dfrac{e_{15}}{C_{44}}\eta & 0 & \left(\varepsilon_{11} + \dfrac{e_{15}^2}{C_{44}}\right)(\xi^2 + \eta^2) \end{bmatrix} \tag{23}$$

$$\mathbf{B} = \begin{bmatrix} A_1\xi^2 + C_{66}\eta^2 & (A_2 + C_{66})\xi\eta & A_4\xi & A_3\xi \\ (A_2 + C_{66})\xi\eta & C_{66}\xi^2 + A_1\eta^2 & A_4\eta & A_3\eta \\ A_4\xi & A_4\eta & -A\varepsilon_{33} & -Ae_{33} \\ A_3\xi & A_3\eta & -Ae_{33} & -AC_{33} \end{bmatrix}. \tag{24}$$

It is noted that the matrices $\mathbf{A}$ and $\mathbf{B}$ are now rendered symmetric (different from $\mathbf{P}$ and $\mathbf{Q}$) by partitioning and redefining the variables. This symmetry shall be utilized to simplify the computational task later on. Following the procedure described in Bahar (1977), we write the solution to (21) as:

$$\bar{\mathbf{R}}_1(z) = \cosh(z\sqrt{\mathbf{G}})\bar{\mathbf{R}}_1(0) + \frac{\mathbf{A}}{\sqrt{\mathbf{H}}}\sinh(z\sqrt{\mathbf{H}})\bar{\mathbf{R}}_2(0)$$

$$\bar{\mathbf{R}}_2(z) = \cosh(z\sqrt{\mathbf{H}})\bar{\mathbf{R}}_2(0) + \frac{\mathbf{B}}{\sqrt{\mathbf{G}}}\sinh(z\sqrt{\mathbf{G}})\bar{\mathbf{R}}_1(0) \tag{25}$$

where

$$\mathbf{G} = \mathbf{AB} \quad \text{and} \quad \mathbf{H} = \mathbf{BA}. \tag{26}$$

We note that the above solution still contains both the matrices **A** and **B**. We shall exploit the properties of the matrices involved. It follows from (26) that

$$\mathbf{G}^{\mathrm{T}} = (\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \mathbf{H} \tag{27}$$

and

$$|\mathbf{G} - \lambda\mathbf{I}| = |\mathbf{H}^{\mathrm{T}} - \lambda\mathbf{I}| = |(\mathbf{H} - \lambda\mathbf{I})^{\mathrm{T}}| = |\mathbf{H} - \lambda\mathbf{I}|$$

for arbitrary λ. Hence, the eigenvalues of **G** are the same as those of **H**. It follows also from (15) that

$$\cosh(z\sqrt{\mathbf{G}}) = (\cosh(z\sqrt{\mathbf{H}}))^{\mathrm{T}} \tag{28}$$

and

$$\frac{1}{\sqrt{\mathbf{H}}}\sinh(z\sqrt{\mathbf{H}}) = \left[\frac{1}{\sqrt{\mathbf{G}}}\sinh(z\sqrt{\mathbf{G}})\right]^{\mathrm{T}}. \tag{29}$$

Therefore, the solution (16) can be rewritten entirely in terms of only one matrix, either **G** or **H**. The solution is now given by

$$\bar{\mathbf{R}}_1(z) = \cosh(z\sqrt{\mathbf{G}})\bar{\mathbf{R}}_1(0) + \mathbf{A}\left[\frac{\sinh(z\sqrt{\mathbf{G}})}{\sqrt{\mathbf{G}}}\right]^{\mathrm{T}}\bar{\mathbf{R}}_2(0)$$

$$\bar{\mathbf{R}}_2(z) = \mathbf{B}\left[\frac{\sinh(z\sqrt{\mathbf{G}})}{\sqrt{\mathbf{G}}}\right]\bar{\mathbf{R}}_1(0) + \cosh(z\sqrt{\mathbf{G}})^{\mathrm{T}}\bar{\mathbf{R}}_2(0) \tag{30}$$

which contains only matrix **G**. The matrix functions

$$\cosh(a\sqrt{\mathbf{G}}) \quad \text{and} \quad \frac{1}{\sqrt{\mathbf{G}}}\sinh(z\sqrt{\mathbf{G}})$$

are expanded into a matrix polynomial as follows:

$$\cosh(z\sqrt{\mathbf{G}}) = \sum_{i=0}^{3} a_i\mathbf{G}^i$$

$$\frac{1}{\sqrt{\mathbf{G}}}\sinh(z\sqrt{\mathbf{G}}) = \sum_{i=0}^{3} b_i\mathbf{G}^i. \tag{31}$$

where no higher powers of **G** are needed on account of the Cayley–Hamilton theorem.

Assuming that matrix **G** takes distinct eigenvalues, say η_0, η_1, η_2 and η_3, the coefficients a_i and b_i in (31) can be determined as follows:

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & \eta_0 & \eta_0^2 & \eta_0^3 \\ 1 & \eta_1 & \eta_1^2 & \eta_1^3 \\ 1 & \eta_2 & \eta_2^2 & \eta_2^3 \\ 1 & \eta_3 & \eta_3^2 & \eta_3^3 \end{bmatrix}^{-1} \begin{Bmatrix} \cosh(z\eta_0) \\ \cosh(z\eta_1) \\ \cosh(z\eta_2) \\ \cosh(z\eta_3) \end{Bmatrix} \tag{32a}$$

$$\begin{Bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \begin{bmatrix} 1 & \eta_0 & \eta_0^2 & \eta_0^3 \\ 1 & \eta_1 & \eta_1^2 & \eta_1^3 \\ 1 & \eta_2 & \eta_2^2 & \eta_2^3 \\ 1 & \eta_3 & \eta_3^2 & \eta_3^3 \end{bmatrix}^{-1} \begin{Bmatrix} \frac{1}{\sqrt{\eta_0}} \sinh(z\eta_0) \\ \frac{1}{\sqrt{\eta_1}} \sinh(z\eta_1) \\ \frac{1}{\sqrt{\eta_2}} \sinh(z\eta_2) \\ \frac{1}{\sqrt{\eta_3}} \sinh(z\eta_3) \end{Bmatrix}. \tag{32b}$$

Once the matrix functions in (31) are determined, the transfer matrix in (30) can readily be obtained. Then boundary conditions on the top and bottom surfaces are introduced to determine the unknown quantities on the boundary. Once all the quantities on the boundary are determined, (30) and (6) can be used to calculate all unknown quantities at arbitrary z.

5. NUMERICAL EXAMPLES

Here, we present a numerical study of a rectangular piezoelectric lamina for which an exact solution is presented above. The material chosen for the lamina is $BaTiO_3$ which has the following material properties (Berlincourt *et al.*, 1964; Shindo *et al.*, 1993): $C_{11} = 16.6$, $C_{12} = 7.7$, $C_{13} = 7.8$, $C_{33} = 16.2$, $C_{44} = 4.3$ (10^{10} Nm^{-2}); $e_{31} = -4.4$, $e_{33} = 18.6$, $e_{15} = 11.6$ (Cm^{-2}); $\varepsilon_{11} = 1.12$, $\varepsilon_{33} = 1.26$ (10^{-8} CVm^{-1}). Although the primary objective of this numerical study is to test the procedure described above, the analysis should also reveal basic tenets of the electro-elastic coupling in the piezoelectric lamina. For the purpose of comparison, a purely elastic lamina with the same elastic properties is also considered (piezoelectric constants are set to zero). It should be remembered that all state variables (three displacement components, electrical potential, three out-of-plane stresses, and transverse electric displacement) are obtained simultaneously in our approach. The in-plane stresses and electric displacements (σ_{xx}, σ_{yy}, σ_{xy}, D_x and D_y) are then obtained from (6). Although the mechanical and electrical loading is considered separately here for transparency, a combined mechanical and electrical loading, if desired, can be treated without any additional difficulty. A sinusoidal distribution is assumed for both the mechanical and electrical loading for the sake of simplicity.

Mechanical loading

A simply supported square lamina ($a = b = 1$ m) of thickness 0.2 (m) is subjected to a mechanical load on the top surface with the following sinusoidal distribution:

$$\sigma_{zz} = \sigma_0 \sin(\pi x/a)\sin(\pi y/b), \tag{33}$$

where $\sigma_0 = 1$ (Nm^{-2}). The field point is chosen at $x/a = 0.75$ and $y/b = 0.25$. Figures 1 and 2 show the variation of U and V, respectively, as a function of the thickness coordinate z where the solid line stands for the piezoelectric lamina and the dotted line for the purely elastic one. The in-plane displacements exhibit linear distribution across the thickness of the lamina. Note that the discrepancy between the two cases is rather small. The transverse displacement W for the two cases is shown in Fig. 3 which demonstrates essentially uniform distribution across the thickness although its magnitudes are quite different between the

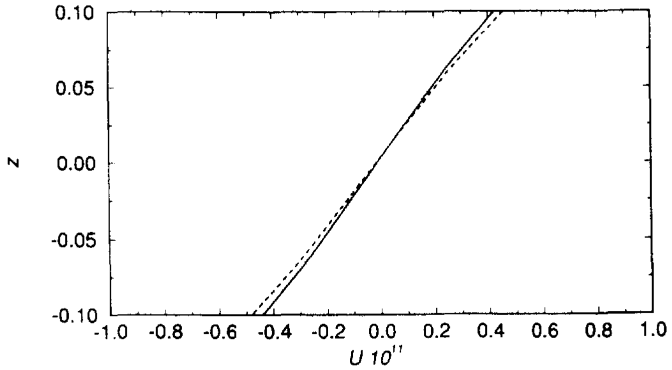

Fig. 1. U vs z.

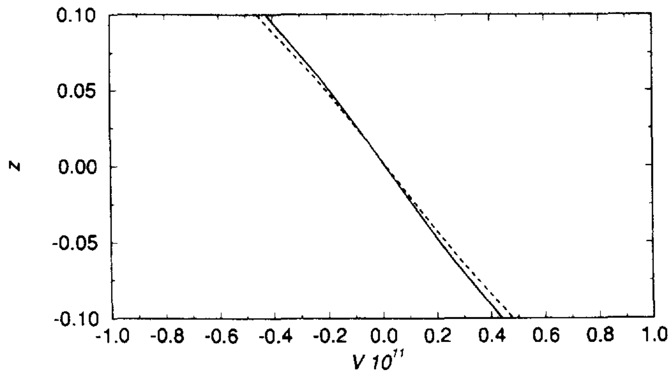

Fig. 2. V vs z.

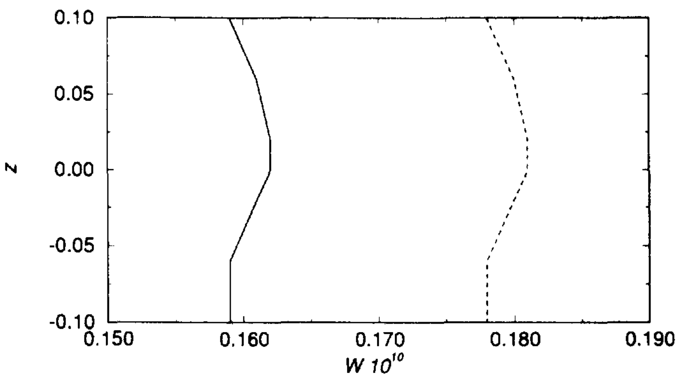

Fig. 3. W vs z.

two cases. Figure 4 shows the electric potential ϕ across the thickness for piezoelectric plate. The potential exhibits a polynomial distribution. Note that the electric potential ϕ is zero in the case of purely elastic plate due to the obvious lack of electroelastic coupling. Figure 5 shows the distribution of the out-of-plane stress σ_{xz} (due to the symmetry of the problem, σ_{yz} exhibits similar distribution). Figure 6 shows the distribution of out-of-plane stress component σ_{zz}. It is interesting to note that all out-of-plane stresses take the same values between the two cases. Figure 7 shows the out-of-plane electric displacement D_z and Fig. 8 shows the in-plane stress σ_{xx} for the two cases. Note that other in-plane stress components σ_{xy} and σ_{yy} exhibit similar distribution as σ_{xx}.

Electric loading

The same plate is loaded electrically on the upper surface by a transverse electric displacement of sinusoidal distribution :

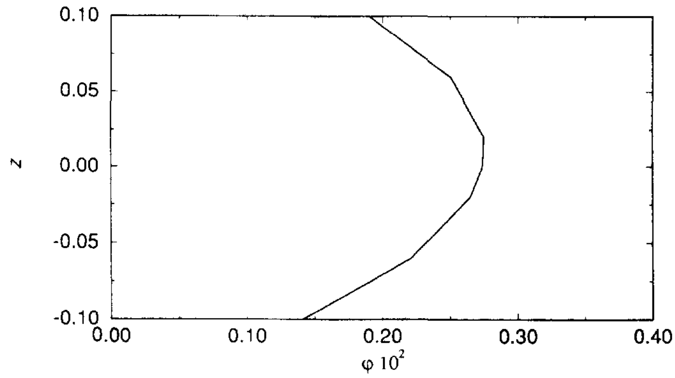

Fig. 4. ϕ vs z.

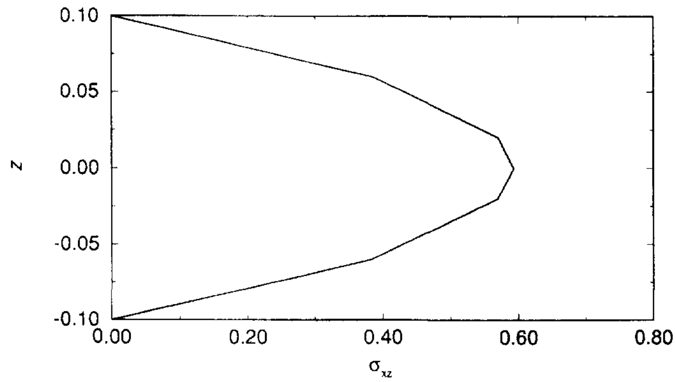

Fig. 5. σ_{xz} vs z.

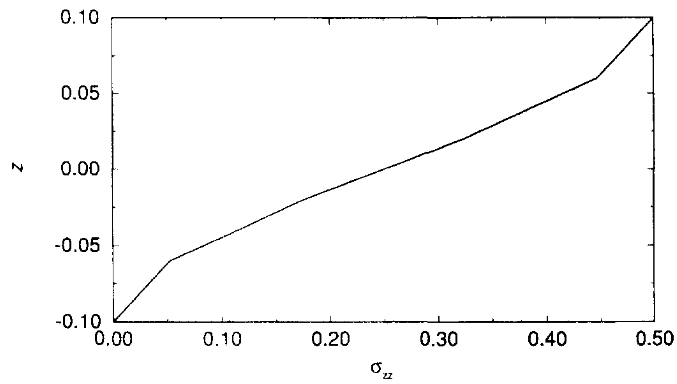

Fig. 6. σ_{zz} vs z.

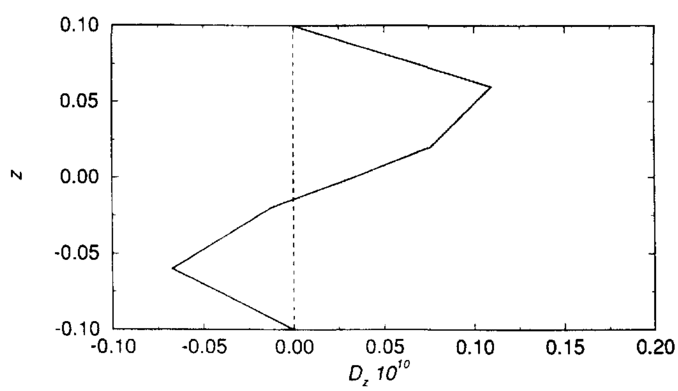

Fig. 7. D_z vs z.

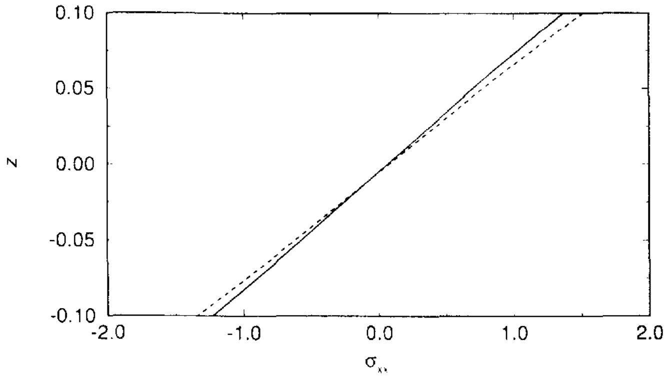

Fig. 8. σ_{xx} vs z.

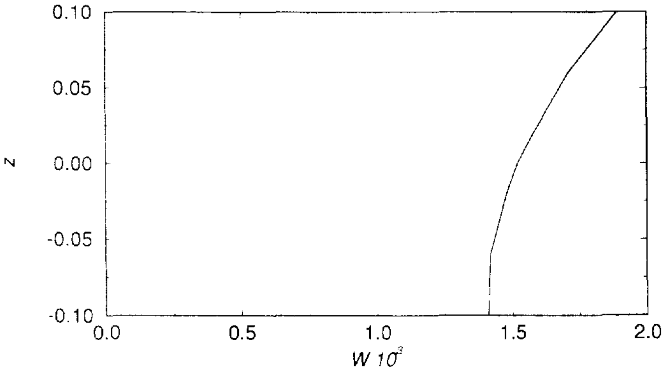

Fig. 9. W vs z.

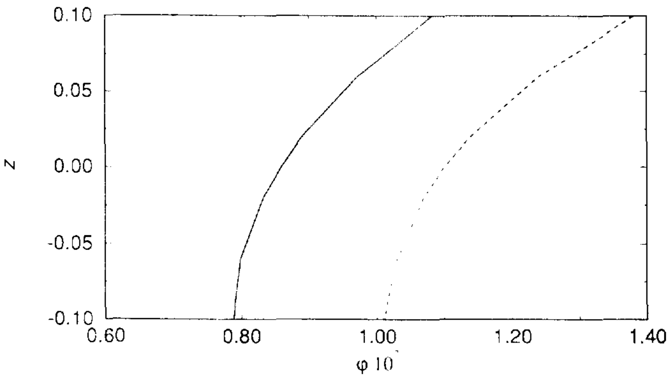

Fig. 10. ϕ vs z.

$$D_z = D_0 \sin(\pi x/a)\sin(\pi y/b) \tag{34}$$

where D_0 is taken as 1 ($\mathrm{C\,m^{-2}}$). The field point is chosen at the same location as the previous example of mechanically loaded plate. Figures 9 and 10 show the distribution of the transverse displacement and electric potential across the thickness of the plate, respectively. Note that there exist significant discrepancies between the two cases. Figure 11 shows the distribution of the out-of-plane stress σ_{yz} for the two cases. Figure 12 shows the distribution of D_z which exhibits no discrepancy between the two cases. The in-plane stress σ_{xx} is shown in Fig. 13. Figure 14 shows the distribution of the in-plane electric displacement D_x which shows no discrepancy between the two cases despite our expectation otherwise (based on the previous example of the mechanically-loaded plate). In order to explain this, let us recall the state equation (6):

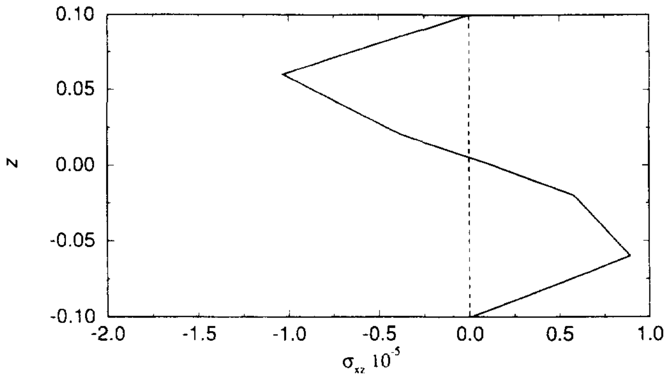

Fig. 11. σ_{xz} vs z.

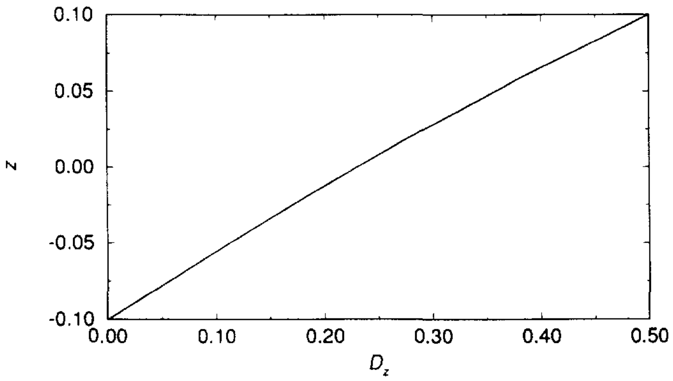

Fig. 12. D_z vs z.

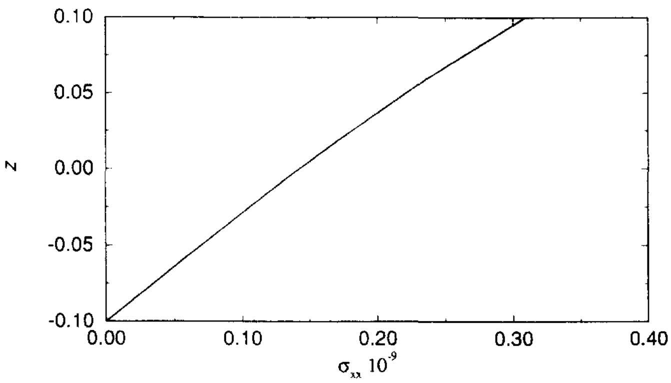

Fig. 13. σ_{xx} vs z.

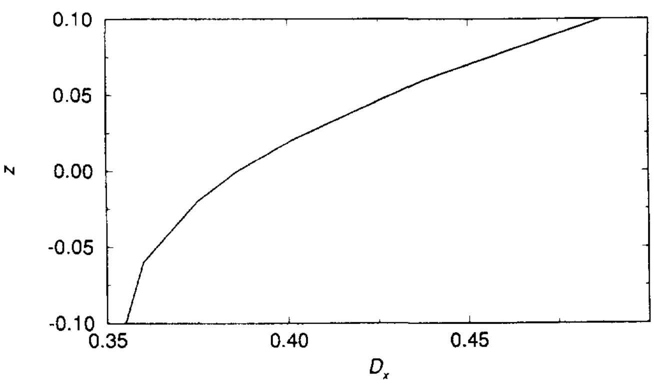

Fig. 14. D_x vs z.

$$D_x = -\varepsilon_{11}\frac{\partial \phi}{\partial x} + \frac{e_{15}}{C_{44}}\sigma_{xz} - \frac{e_{15}^2}{C_{44}}\frac{\partial \phi}{\partial x}, \tag{35}$$

where the first term on the right-hand-side of (35) is the electric displacement from the uncoupled electrostatics and the next two terms represent the difference between the two cases arising from the electroelastic interaction. For the material chosen for this example, e_{15}/C_{44} is $O(10^{-10})$ and σ_{xz} is $O(10^{-5})$. Therefore, the second term in the right-hand-side of (35) has very little contribution. The coefficient e_{15}^2/C_{44} of the third term is also very small due to the same reason, whereas the value of $\partial\phi/\partial x$ is very close between the two cases. Therefore, there is no noticeable discrepancy between the two cases.

6. CONCLUDING REMARKS

A state-space-based solution method was developed to study electroelastic responses of a piezoelectric lamina in an attempt to develop an efficient analytical technique for electromechanical analysis of laminated piezoelectric structures. The state space methodology developed was employed to obtain an exact solution for a rectangular piezoelectric lamina with simple supports in the form of infinite series. A square piezoelectric plate made of $BaTiO_3$ is considered as a numerical example and electromechanical responses were compared to those of a purely elastic one. It is noted that although only one particular class of crystal symmetry (namely, 6mm class) is considered, this study entails a reasonable degree of generality since most man-made piezoelectric materials commonly used in laminated electromechanical devices fall into this symmetry class and full electromechanical coupling as well as material anisotropy are preserved. The most interesting result obtained from the analysis was that the out-of-plane stresses and electric displacement were not influenced by the electroelastic coupling. However, further studies are required for more definitive conclusions on this issue. The analytical approach developed herein can readily be extended for the study of multilayered laminated structures with general interlayer and boundary conditions. Since the transfer matrix of a layer is known, a global transfer matrix for a layered structure can be assembled by incorporating the interlayer contact conditions and boundary conditions. The order of the global transfer matrix does not depend on the number of layers since the transfer matrix is multiplicative in nature for some common interlayer contact conditions (e.g. perfectly bonded contact). For other general interlayer contact or boundary conditions, such as partially bonded contacts or Winkler mattress, the global transfer matrix can be constructed by adopting the generalized procedure suggested in Lee and Jiang (1994b) for purely elastic layered media.

Acknowledgements—The material presented in this study is based upon work supported in part by the National Science Foundation under grant MSS-9313216. The support is gratefully acknowledged.

REFERENCES

Bahar, L. Y. (1977). Discussion on "A mixed method in elasticity." *J. Appl. Mech.* **44**, 790–791.

Berlincourt, D. A. *et al.* (1964). Piezoelectric and piezoceramic materials and their function in transucers. In *Physical Acoustics* (Edited by W. Mason), Vol. I-A. Academic Press, New York.

Chen, T. (1993). Green's functions and the non-uniform transformation problem in a piezoelectric medium. *Mech. Res. Commun.* **20**, 271–278.

Dunn, M. L. (1994). Electroelastic Green's functions for transversely isotropic piezoelectric media and their application to the solution of inclusion and inhomogeneity problems. *Int. J. Engng Sci.* **32**, 119–131.

Eringen, A. C. and Maugin, G. A. (1991). *Electrodynamics of Continua I*. Springer, Berlin.

Gantmacher, F. R. (1960). *The Theory of Matrix*. Chelsea, New York.

Iyengar, K. T. S. R. and Pandya, S. K. (1983). Analysis of orthotropic rectangular thick plates. *Fiber Sci. Tech.* **18**, 19–36.

Jiang, L. Z. and Lee, J. S. (1993). Discussion on "On the modeling of piezoelectric laminated structures". *Mech. Res. Commun.* **20**, 437.

Lee, J. S., Maugin, G. A. and Shindo, Y. (Eds) (1993). *Mechanics of Electromagnetic Materials and Structures*, AMD-Vol. 161. ASME, New York.

Lee, J. S. and Jiang, L. Z. (1994a). A boundary integral formulation and 2D fundamental solution for piezoelastic media. *Mech. Res. Commun.* **21**, 47–54.

Lee, J. S. and Jiang, L. Z. (1994b). Axisymmetric analysis of multi-layered transversely isotropic elastic media with general interlayer and support conditions. *Struct. Engng Mech.* **2,** 49–62.

Nye, J. F. (1976). *Physical Properties of Crystal and Their Representation by Tensors and Matrix*. Oxford University Press, Oxford.

Pak, Y. E. (1990). Crack extension force in a piezoelectric material. *J. Appl. Mech.* **57,** 647–653.

Rao, S. S. and Sunar, M. (1994). Piezoelectricity and its use in disturbance sensing and control of flexible structures: A survey. *Appl. Mech. Rev.* **47,** 113–123.

Shindo, Y., Ozawa, E. and Nowacki, J. P. (1990). Singular stress and electric fields of a cracked piezoelectric strip. *Int. J. Appl. Elect. Mat.* **1,** 77–87.

Sosa, H. (1992). On the modeling of piezoelectric laminated structures. *Mech. Res. Commun.* **19,** 541–546.

Sosa, H. and Castro, M. A. (1994). On the concentrated loads at the boundary of a piezoelectric half-plane. *J. Mech. Phys. Solids* **42,** 1105–1122.

Steele, R. C. and Kim, Y. Y. (1992). Modified mixed variational principle and the state vector equation for elastic bodies and shells of revolution. *J. Appl. Mech.* **59,** 587–595.

Tiersten, H. F. (1969). *Linear Piezoelectric Plate Vibrations*. Plenum Press, New York.

Tzou, H. S. and Anderson, G. L. (1992). *Intelligent Structural Systems*. Kluwer, The Netherlands.

Vlasov, V. Z. and Leontev, N. N. (1966). *Beams, Plates and Shells on Elastic Foundations*. NASA TTF-357, TT65-50135.

Wang, B. (1992). Three-dimensional analysis of an ellipsoidal inclusion in a piezoelectric material. *Int. J. Solids Struct.* **29,** 293–308.